Chapter 9

Multiple constraints and compound objectives

9.1 Introduction and synopsis

Most decisions you make in life involve trade-offs. Sometimes the trade-off is to cope with conflicting constraints: I must pay this bill but I must also pay that one — you pay the one which is most pressing. At other times the trade-off is to balance divergent objectives: I want to be rich but I also want to be happy — and resolving this is harder since you must balance the two, and wealth is not measured in the same units as happiness.

So it is with selecting materials. Commonly, the selection must satisfy several, often conflicting, constraints. In the design of an aircraft wing-spar, weight must be minimized, with constraints on stiffness, fatigue strength, toughness and geometry. In the design of a disposable hot-drink cup, cost is what matters; it must be minimized subject to constraints on stiffness, strength and thermal conductivity, though painful experience suggests that designers sometimes neglect the last. In this class of problem there is one design objective (minimization of weight or of cost) with many constraints. Nature being what it is, the choice of material which best satisfies one constraint will not usually be that which best meets the others.

A second class of problem involves divergent objectives, and here the conflict is more severe. The designer charged with selecting a material for a wing-spar that must be both as light *and* as cheap as possible faces an obvious difficulty: the lightest material will certainly not be the cheapest, and vice versa. To make any progress, the designer needs a way of trading off weight against cost. Strategies for dealing with both classes of problem are summarized in Figure 9.1 on which we now expand.

There are a number of quick although subjective ways of dealing with conflicting constraints and objectives: the *sequential index* method, the *method of weight-factors*, and methods employing *fuzzy logic*. They are a good way of getting into the problem, so to speak, but their limitations must be recognized. Subjectivity is eliminated by employing the *active constraint method* to resolve conflicting constraints, and by combining objectives, using *exchange constants*, into a single *value function*.

We use the beam as an example, since it is now familiar. For simplicity we omit shape (or set all shape factorrs equal *to* 1); reintroducing it is straightforward.

9.2 Selection by successive application of property limits and indices

Suppose you want a material for a light beam (the objective) which is both stiff (constraint 1) and strong (constraint 2), as in Figure 9.2. You could choose materials with high modulus E for

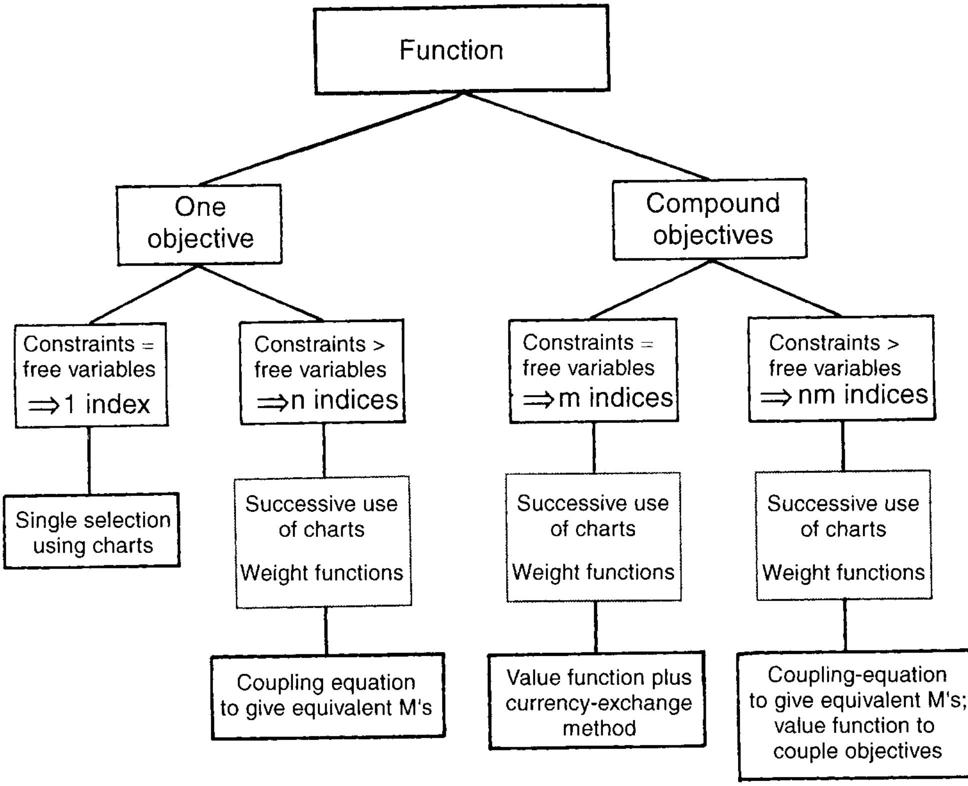

Fig. 9.1 The procedures for dealing with multiple constraints and compound objectives.

stiffness, and then the subset of these which have high elastic limits σ_y for strength, and the subset of those which have low density ρ for light weight. Some selection systems work that way, but it is not a good idea because there is no guidance in deciding the relative importance of the limits on E, σ_y and ρ.

A better idea: first select the subset of materials which is light and stiff (index $E^{1/2}/\rho$), then the subset which is light and strong (index $\sigma_y^{2/3}/\rho$), and then seek the common members of the two subsets. Then you have combined some of the properties in the right way.

Put more formally: an objective function is identified; each constraint is used in turn to eliminate the free variable, temporarily ignoring the others, giving a set of material-indices (which we shall call M_i) which are ranked according to the importance, in your judgement, of the constraints from which they arise. Then a subset of materials is identified which has large values of the first index, M_1, either by direct calculation or by using the appropriate selection chart. The subset is left large enough to allow the remaining constraints to be applied to it.

The second index M_2 is now applied, identifying a second subset of materials. Common members of the two subsets are identified and ranked according to their success in maximizing the two indices. It will be necessary to iterate, narrowing the subset controlled by the hard constraints, broadening that of the softer ones. The procedure can be repeated, using further constraints, as often as needed provided the initial subsets are not made too small. The same method can be applied to multiple objectives.

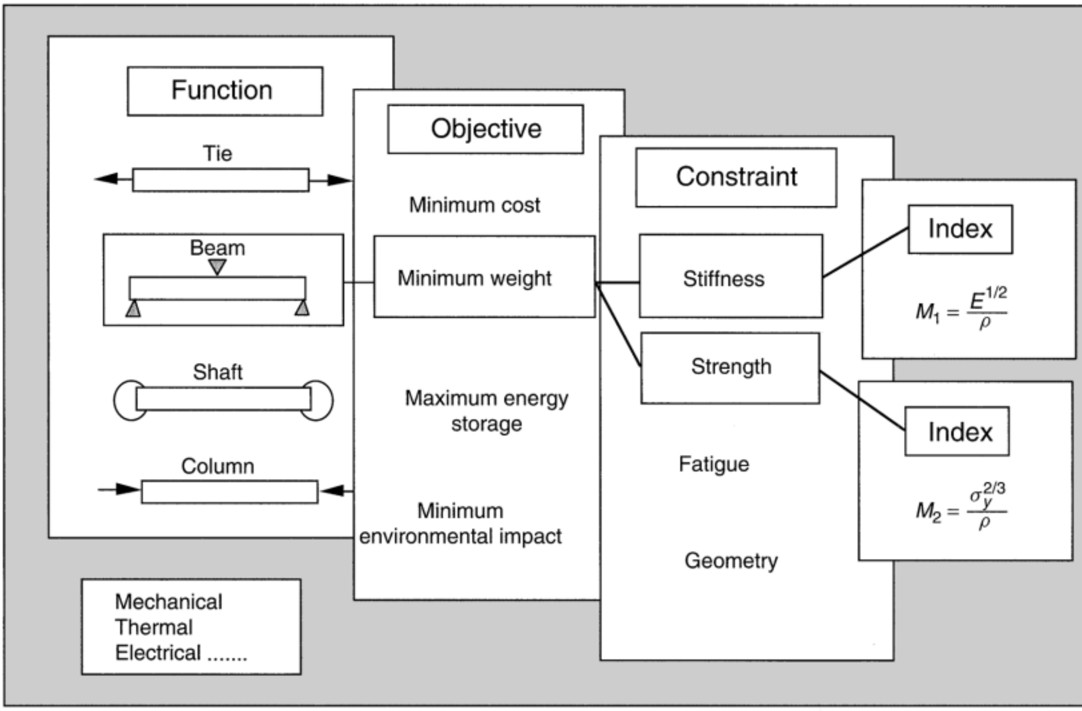

Fig. 9.2 One objective (here, minimizing mass) and two constraints (stiffness and strength) lead to two indices.

This approach is quick (particularly if it is carried out using computer-based methods*), and it is a good way of getting a feel for the way a selection exercise is likely to evolve. But it is far from perfect, because it involves judgement in placing the boundaries of the subsets. Making judgements is a part of materials selection — the context of any real design is sufficiently complex that expert judgmental skills is always needed. But there are problems with the judgements involved in the successive use of indices. The greatest is that of avoiding subjectivity. Two informed people applying the same method can get radically different results because of the sensitivity of the outcome to the way the judgements are applied.

9.3 The method of weight-factors

Weight-factors express judgements in a more formal way. They provide a way of dealing with quantifiable properties (like E, or ρ, or $E^{1/2}/\rho$) and also with properties which are difficult to quantify, like corrosion and wear.

The method, applied to material selection, works like this. The key properties or indices are identified and their values M_i are tabulated for promising candidates. Since their absolute values can differ widely and depend on the units in which they are measured, each is first scaled by dividing it by the largest index of its group, $(M_i)_{max}$, so that the largest, after scaling, has the value 1. Each is

* See, for example, the CMS selection software marketed by Granta Design (1995).

then multiplied by a weight-factor, w_i, which expresses its relative importance for the performance of the component, to give a weighted index W_i:

$$W_i = w_i \frac{M_i}{(M_i)_{max}} \tag{9.1}$$

For properties that are not readily expressed as numerical values, such as weldability or wear resistance, rankings such as A to E are expressed instead by a numeric rating, A = 5 (very good) to E = 1 (very bad) and then, as before, dividing by the highest rating value. For properties that are to be minimized, like corrosion rate, the scaling uses the minimum value $(M_i)_{min}$, expressed in the form

$$W_i = w_i \frac{(M_i)_{min}}{M_i}$$

The weight-factors w_i are chosen such that they add up to 1, that is: $w_i < 1$ and $\Sigma w_i = 1$. There are numerous schemes for assigning their values (see Further Reading: Weight factors). All require, in varying degrees, the use of judgement. The most important property or index is given the largest w, the second most important, the second largest and so on. The W_i are calculated from equation (9.1) and summed. The best selection is the material with the largest value of the sum

$$W = \sum_i W_i = \sum_i w_i \frac{M_i}{(M_i)_{max}} \tag{9.2}$$

But there are problems with the method, some obvious (like that of assigning values for the weight factors), some more subtle*. Here is an example: the selection of a material for a light beam which must meet constraints on both stiffness (index $M_1 = E^{1/2}/\rho$) and strength (index $M_2 = \sigma_y^{2/3}/\rho$). The values of these indices are tabulated for four materials in Table 9.1. Stiffness, in our judgement, is more important than strength, so we assign it the weight factor

$$w_1 = 0.7$$

That for strength is then

$$w_2 = 0.3$$

Normalize the index values (as in equation (9.1)) and sum them (equation (9.2)) to give W. The second last column of Table 9.1 shows the result: beryllium wins easily; Ti-6-4 comes second, 6061 aluminium third. But observe what happens if beryllium (which can be toxic) is omitted from the selection, leaving only the first three materials. The same procedure now leads to the values of W in the last column: 6061 aluminium wins, Ti-6-4 is second. Removing one, non-viable, material

Table 9.1 Example of use of weight factors

Material	ρ *Mg/m³*	E *GPa*	σ_y *MPa*	$E^{1/2}/\rho$	$\sigma_y^{2/3}/\rho$	W *(inc. Be)*	W *(excl. Be)*
1020 Steel	7.85	205	320	1.82	6.0	0.24	0.52
6061 Al (T4)	2.7	70	120	3.1	9.0	0.39	**0.86**
Ti-6-4	4.4	115	950	2.4	**17.1**	0.48	0.84
Beryllium	1.86	300	170	**9.3**	16.5	**0.98**	—

* For a fuller discussion see de Neufville and Stafford (1971) or Field and de Neufville (1988).

from the selection has reversed the ranking of those which remain. Even if the weight factors could be chosen with accuracy, this dependence of the outcome on the population from which the choice is made is disturbing. The method is inherently unstable, sensitive to irrelevant alternatives.

The most important factor, of course, is the set of values chosen for the weight-factors. The schemes for selecting them are structured to minimize subjectivity, but an element of personal judgement inevitably remains. The method gives pointers, but is not a rigorous tool.

9.4 Methods employing fuzzy logic

Fuzzy logic takes weight-factors one step further. Figure 9.3 at the upper left, shows the probability P(*R*) of a material having a property or index-value in a given range of *R*. Here the property has a well-defined range for each of the four materials A, B, C and D (the values are *crisp* in the terminology of the field). The selection criterion, shown at the top right, identifies the range of *R* which is sought for the properties, and it is *fuzzy*, that is to say, it has a well-defined *core* defining the ideal range sought for the property, with a wider *base*, extending the range to include boundary regions in which the value of the property or index is allowable, but with decreasing acceptability as the edges of the base are approached.

The superposition of the two figures, shown at the centre of Figure 9.3, illustrates a single selection stage. Desirability is measured by the product P(*R*)S(*R*). Here material B is fully acceptable — it acquires a weight of 1. Material A is acceptable but with a lower weight, here 0.5; C is acceptable with a weight of roughly 0.25, and D is unacceptable — it has a weight of 0. At the end

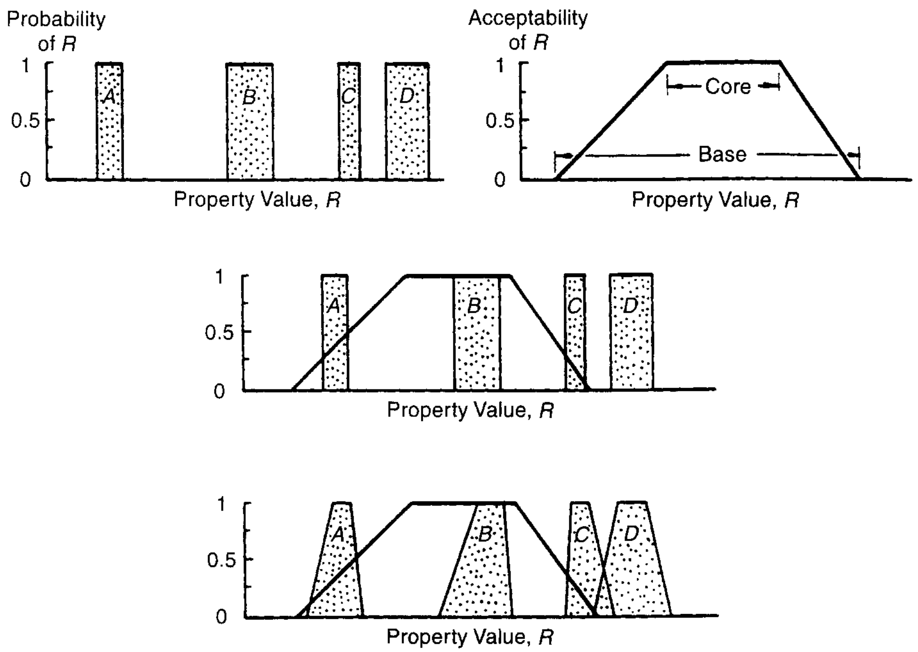

Fig. 9.3 Fuzzy selection methods. Sharply-defined properties and a fuzzy selection criterion, shown at (a), are combined to give weight-factors for each material at (b). The properties themselves can be given fuzzy ranges, as shown at (c).

of the first selection stage, each material in the database has one weight-factor associated with it. The procedure is repeated for successive stages, which could include indices derived from other constraints or objectives. The weights for each material are aggregated — by multiplying them together, for instance — to give each a super-weight with a value between 0 (totally unacceptable) to 1 (fully acceptable by all criteria). The method can be refined further by giving fuzzy boundaries to the material properties or indices as well as to the selection criteria, as illustrated in the lower part of Figure 9.3. Techniques exist to choose the positions of the cores and the bases, but despite the sophistication the basic problem remains: the selection of the ranges S(R) is a matter of judgement.

Successive selection, weight factors and fuzzy methods all have merit when more rigorous analysis, of the sort described next, is impractical. And they can be fast. They are a good first step. But if you really want to identify the best material for a complex design, you need to go further. Ways of doing that come next.

9.5 Systematic methods for multiple constraints

Commonly, the specification of a component results in a design with multiple constraints, as in the second column of Figure 9.1. Here the *active constraint method* is the best way forward. It is systematic — it removes the dependence on judgement. The idea is simple enough. Identify the most restrictive constraint. Base the design on that. Since it is the most restrictive, all other constraints will automatically be satisfied.

The method is best illustrated through an example. We stay with that of the light, stiff, strong beam. For simplicity, we leave out shape (including it involves no new ideas). The objective function is

$$m = A\ell\rho \tag{9.3}$$

where $A = t^2$ is the area of the cross-section. The first constraint is that on stiffness, S

$$S = \frac{C_1 EI}{\ell^3} \tag{9.4}$$

with $I = t^4/12$ and $C_1 = 48$ for the mode of loading shown in Figure 9.4; the other variables have the same definitions as in Chapter 5. Using this to eliminate A in equation (9.3) gives the mass of the beam which will just provide this stiffness S (equation (5.10), repeated here):

$$m_1 = \left(\frac{12\,S}{C_1\,\ell}\right)^{1/2} \ell^3 \left[\frac{\rho}{E^{1/2}}\right] \tag{9.5}$$

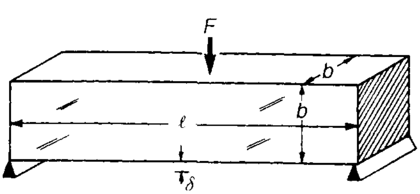

Fig. 9.4 A square-section beam loaded in bending. It has a second moment of area $I = t^4/12$. It must have a prescribed stiffness S and strength F_f, and be as light as possible.

The second constraint is that on strength. The collapse load of a beam is

$$F_f = C_2 \frac{I\sigma_y}{y_m \ell} \tag{9.6}$$

where $C_2 = 4$ and $y_m = t/2$ for the configuration shown in the figure. Using this instead of equation (9.4) to eliminate A in equation (9.3) gives the mass of the beam which will just support the load F_f:

$$m_2 = \left(\frac{6}{C_2}\frac{F_f}{\ell^2}\right)^{2/3} \ell^3 \left[\frac{\rho}{\sigma_y^{2/3}}\right] \tag{9.7}$$

More constraints simply lead to more such equations for m.

If the beam is to meet both constraints, its weight is determined by the larger of m_1 and m_2; if there are i constraints, then it is determined by the largest of all the m_i. Define $\tilde{m}$ as

$$\boxed{\tilde{m} = \max(m_1, m_2, m_3, \ldots)} \tag{9.8}$$

The best choice is that of the material with the smallest value of $\tilde{m}$. It is the lightest one that meets or exceeds all the constraints.

That is it. Now the ways to use it.

The analytical method

Table 9.2 illustrates the use of the method to select a material for a light, stiff, strong beam of length ℓ, stiffness S and collapse load F_f with the values

$$\ell = 1\,\text{m} \qquad S = 10^6\,\text{N/m} \qquad F_f = 2 \times 10^4\,\text{N}$$

Substituting these values and the material properties shown in the table into equations (9.5) and (9.7) gives the values for m_1 and m_2 shown in the table. The last column shows $\tilde{m}$ calculated from equation (9.8). For these design requirements Ti-6-4 is emphatically the best choice: it allows the lightest beam which satisfies both constraints.

The best choice depends on the details of the design requirements; a change in the prescribed values of S and F_f alters the selection. This is an example of the power of using a systematic method: it leads to a selection which does not rely on judgement; two people using it independently will reach exactly the same conclusion. And the method is robust: the outcome is not influenced by irrelevant alternatives. It can be generalized and presented on selection charts (allowing a clear graphical display even when the number of materials is large) as described next.

Table 9.2 Selection of a material for a light, stiff, strong beam

Material	ρ *kg/m³*	*E* *GPa*	σ_y *MPa*	m_1 *kg*	m_2 *kg*	$\tilde{m}$ *kg*
1020 Steel	7850	205	320	8.7	**16.2**	16.2
6061 Al	2700	70	120	5.1	**10.7**	10.7
Ti-6-4	4400	115	950	**6.5**	4.4	**6.5**

The graphical method

Stated more formally, the steps of the example in the last section were these.

(a) Express the objective as an equation, here equation (9.3).
(b) Eliminate the free variable using each constraint in turn, giving sets of performance equations (objective functions) with the form.

$$P_1 = f_1(F)g_1(G)M_1 \qquad 9.9\text{(a)}$$

$$P_2 = f_2(F)g_2(G)M_2 \qquad 9.9\text{(b)}$$

$$P_3 = f_3(F)\ldots \text{ etc.}$$

where f and g are expressions containing the functional requirements F and geometry G, and M_1 and M_2 are material indices. In the example, these are equations (9.5) and (9.7).
(c) If the first constraint is the most restrictive (that is, it is the *active* constraint), the performance is given by equation (9.9a), and this is maximized by seeking materials with the best values of M_1 ($E^{1/2}/\rho$ in the example). When the second constraint is the active one, the performance equation is given by equation (9.9b) and the highest values of M_2 (here, $\sigma_y^{2/3}/\rho$) must be sought. And so on.

In the example above, performance was measured by the mass m. The selection was made by evaluating m_1 and m_2 and comparing them to identify the active constraint, which, as Table 9.2 shows, depends on the material itself. The same thing can be achieved graphically for two constraints (and more if repeated), with the additional benefit that it displays, in a single picture, the active constraint and the best material choice even when the number of materials is large. It works like this.

Imagine a chart with axes of M_1 and M_2, as in Figure 9.5. It can be divided into two domains in each of which one constraint is active, the other inactive. The switch of active constraint lies at the boundary between the two regimes; it is the line along which the equations (9.9a) and (9.9b)

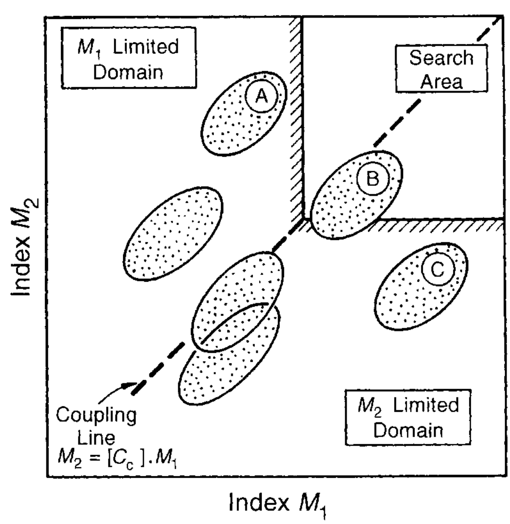

Fig. 9.5 A chart with two indices as axes, showing a box-shaped contour of constant performance. The corner of the box lies on the coupling line. The best choices are the materials which lie in the box which lies highest up the coupling line.

are equal. Equating them and rearranging gives:

$$M_2 = \left[\frac{f_1(F)g_1(G)}{f_2(F)g_2(G)}\right] M_1 \tag{9.10}$$

or

$$M_2 = [C_c]M_1 \tag{9.11}$$

This equation couples the two indices M_1 and M_2; we shall call it the *coupling equation*. The quantity in square brackets — the *coupling constant*, C_c — is fixed by the specification of the design. Materials with M_2/M_1 larger than this value lie in the M_1-limited domain. For these, the first constraint is active and performance limited by equation (9.9a) and thus by M_1. Those with M_2/M_1 smaller than C_c lie in the M_2-limited domain; the second constraint is active and performance limited by equation (9.9b) and thus by M_2. It is these conditions which identify the box-shaped search region shown in Figure 9.5. The corner of the box lies on the coupling line (equation (9.11)); moving the box up the coupling line narrows the selection, identifying the subset of materials which maximize the performance while simultaneously meeting both constraints. Change in the value of the functional requirements F or the geometry G changes the coupling constant, shifts the line, moves the box and changes the selection.

Taking the example earlier in this section and equating m_1 to m_2 gives:

$$M_2 = \left[\left(\frac{6F_f}{C_2\ell^2}\right)^{2/3}\left(\frac{C_1}{12}\frac{\ell}{S}\right)^{1/2}\right] M_1 \tag{9.12}$$

with $M_1 = E^{1/2}/\rho$ and $M_2 = \sigma_y^{2/3}/\rho$. The quantity in square brackets is the coupling constant. It depends on the values of stiffness S and collapse load F_f, or more specifically, on the two structural loading coefficients* S/ℓ and F_f/ℓ^2. They define the position of the coupling line, and thus the selection.

Worked examples are given in Chapter 10.

9.6 Compound objectives, exchange constants and value-functions

Cost, price and utility

Almost always, a design requires the coupled optimization of two or more measures of performance; it has *compound objectives* (Figure 9.1, third column and Figure 9.6). The designer's objective for a performance bicycle might be to make it as light as possible; his marketing manager might insist that it be as cheap as possible. The owner's objective in insulating his house might be to minimize heat loss, but legislation might require that the environmental impact of the blowing agent contained in the insulation be minimized instead. These examples reveal the difficulties: the individual objectives conflict, requiring that a compromise be sought; and in seeking it, how is weight to be compared with cost, or heat flow with environmental impact? Unlike the Ps of the last section, each is measured in different units; they are incommensurate. As mentioned earlier, the judgement-based methods described earlier in this chapter can be used. The 'successive selection' procedure using the charts

* See Section 5.5 for discussion of structural loading coefficients

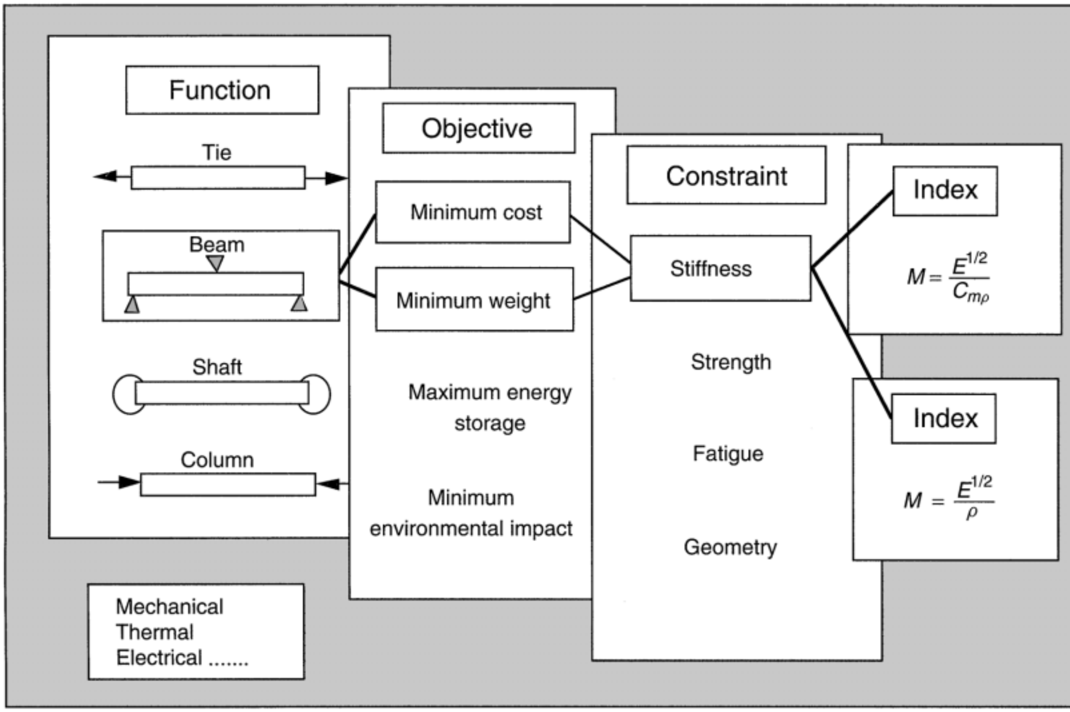

Fig. 9.6 Two objectives (here, minimizing mass and cost) and one constraint (stiffness) lead to two indices.

('first choose the subset of materials which minimizes mass then the subset which minimizes cost, then seek the common members of the two subsets'), and the refinements of it by applying weight-factors or fuzzy logic lead to a selection, but because dissimilar quantities are being compared, the reliance on judgement and the attendant uncertainty is greater than before.

The problem could be overcome if we had a way of relating mass to cost, or energy to environmental impact. With this information a 'compound-objective' or *value function* can be formulated in which the two objectives are properly coupled. A method based on this idea is developed next. To do so, we require *exchange constants* between the objectives which, like exchange-rates between currencies, allows them to be expressed in the same units — in a common currency, so to speak. Any one of those just listed — mass, cost, energy or environmental impact — could be used as the common measure, but the obvious one is cost. Then the exchange constant is given the symbol $E^{\$}$.

First, some definitions. A product has a *cost*, C; it is the sum of the costs to the manufacturer of materials, manufacture and distribution. To the consumer, the product has a *utility* U, a measure, in his or her mind, of the worth of the product. The consumer will be happy to purchase the product if the *price*, P, is less than U; and provided P is greater than C, the manufacturer will be happy too. This desirable state of affairs is summed up by

$$C < P < U \tag{9.13}$$

Exchange any two terms in this equation, and someone is unhappy. The point is that utility is not the same as cost. In some situations a given product can have a high utility, in others it is worthless, even though the cost has not changed. More specific examples in a moment.

Value functions and exchange constants

First, a formal definition; then examples.

A design requires that several (i) objectives must be met. Each objective relates to a performance characteristic P_i with the general form of equations (9.9). The first of these might (for example) describe the mass of the component; the second, the energy consumed in making it; the third, its cost. Ideally we would like to minimize all three, but the cheapest is not the lightest or the most energy efficient; minimizing one does not minimize the others.

To overcome this, define a value function, V, such that

$$V = E_1^{\$}P_1 + E_2^{\$}P_2 + E_3^{\$}P_3 \cdots \tag{9.14}$$

The quantities $E_1^{\$}$, $E_2^{\$}$, etc. are the exchange constants. They convert performances P_i (here with units of kg, MJ and \$) into value (measured in \$, say). Differentiating equation (9.14) gives

$$E_1^{\$} = \left(\frac{\partial V}{\partial P_1}\right)_{P_2,P_3\ldots} \tag{9.14a}$$

$$E_2^{\$} = \left(\frac{\partial V}{\partial P_2}\right)_{P_1,P_3\ldots} \tag{9.14b}$$

and so on. If P_1 is the mass of the component, then $E_1^{\$}$ is the change in value associated with unit change in mass. If P_2 is the energy content, then $E_2^{\$}$ is the change in value associated with unit change in energy content. And if P_3 is the cost of the component, then $E_3^{\$} = -1$ because unit increase in cost give unit decrease in value.

The value of the exchange constant depends on the application. Its value is influenced by many factors, some of them based on sound engineering reasoning, some on market forces, and still others on perceived value. Approximate values for the exchange constant relating mass and cost are listed in Table 9.3. Their values are negative because an increase in mass leads, in these applications, to a decrease in value. In a space-vehicle the value of a mass reduction is high; that of the same mass reduction in a family car is much lower. The ranges given in the top part of the table are related to their applications. They can be estimated approximately in various ways. The cost of launching a payload into space lies in the range \$3000 to \$10 000/kg; a reduction of 1 kg in the weight of the launch structure would allow a corresponding increase in payload, giving the value-range in the table. Similar arguments based on increased payload or decreased fuel consumption give the values shown for civil aircraft, commercial trucks and automobiles. The values change slowly with time, reflecting changes in fuel costs, legislation to increase fuel economy and such like. Special

Table 9.3 Exchange constant for mass saving in transport systems

Transport system	*Exchange constant* $E^{\$}$ *(US\$/kg)*
	(note negative sign)
Family car (based on fuel saving)	−0.5 to −1.5
Truck (based on payload)	−5 to −10
Civil aircraft (based on payload)	−100 to −500
Military aircraft (performance, payload)	−500 to −2000
Space vehicle (based on payload)	−3000 to −10 000
Bicycle (based largely on perceived value)	−80 to −2000

circumstances can change them dramatically — an aero-engine builder who has guaranteed a certain power/weight ratio for his engine, may be willing to pay more than $1000 to save a kilogram if it is the only way in which the guarantee can be met — but we shall ignore these and stay with the more usual situations.

The change in value associated with a unit change in energy consumption has an even larger range (Table 9.4). Here the exchange constant is the change in value associated with unit increase in the energy consumed, and that is simply the cost of energy (with the sign reversed, since an increase in energy consumption, all other factors held constant, gives a decrease value). The exchange constant depends on the form in which the energy is provided, the country in which it is purchased, and — in the case of electricity — the time of day. The example of electricity illustrates just how great can be the variations in exchange constant: grid power for industrial use costs about $0.02/MJ (1 MJ is 3.6 kWh); energy in the form of an AA battery for your walkman costs more than 1000 times more; the energy source in your watch cost you, per MJ, 100 times more than that. But you pay it, because, on your scale of values, it is worth it.

These values for the exchange constant are based on engineering criteria. More difficult to assess are those based on perceived value. That for the weight/cost trade-off for a bicycle is an example. To the enthusiast, a lighter bike is a better bike. Figure 9.7 shows just how much the cyclist values reduction in weight. The tangents give the exchange constant: it ranges from $80/kg to $2000/kg, depending on the mass: the exchange constant depends on the value of the performance characteristic, here mass. Over any small part of the curve it can be linearized giving the tangents, and this is usually acceptable; but if large changes of mass become possible, this dependence must be included (by expressing $E^\$$ as $E^\$(m)$). Does it make sense for the ordinary cyclist to pay $2000 to reduce the mass of the bike by 1 kg when, by dieting, he could reduce the mass of the system (himself plus the bike) by more without spending a penny? Possibly. But mostly it is perceived value. Advertising aims to increase the perceived value of a product, increasing its value without increasing its cost. It influences the exchange constants for family cars and it is the driver for the development of titanium watches, carbon fibre spectacle frames and much more. Perceived, rather than rational, values are frequently associated with the choice of material for sports equipment. And they are harder to pin down.

Determining exchange constants

Exchange constants based on engineering criteria are determined by analysing the economics of the way in which each performance characteristic changes the life-cost of the product. Simple

Table 9.4 Approximate exchange constant of energy

Energy source	*Exchange constant* $E^\$$ *(US\$/MJ)*
Coal	−0.003 to −0.006
Oil	−0.007 to −0.012
Gas	−0.003 to −0.005
Gasoline (US)	−0.012 to −0.015
Gasoline (Europe)	−0.03 to −0.04
Electricity (national grid, US)	−0.02 to −0.03
Electricity (national grid, Europe)	−0.03 to −0.04
Electricity (lead-acid battery, 1000 recharges)	−0.1 to −0.3
Electricity (alkaline AA battery)	−35 to −150
Electricity (silver oxide battery)	−1000 to −3500

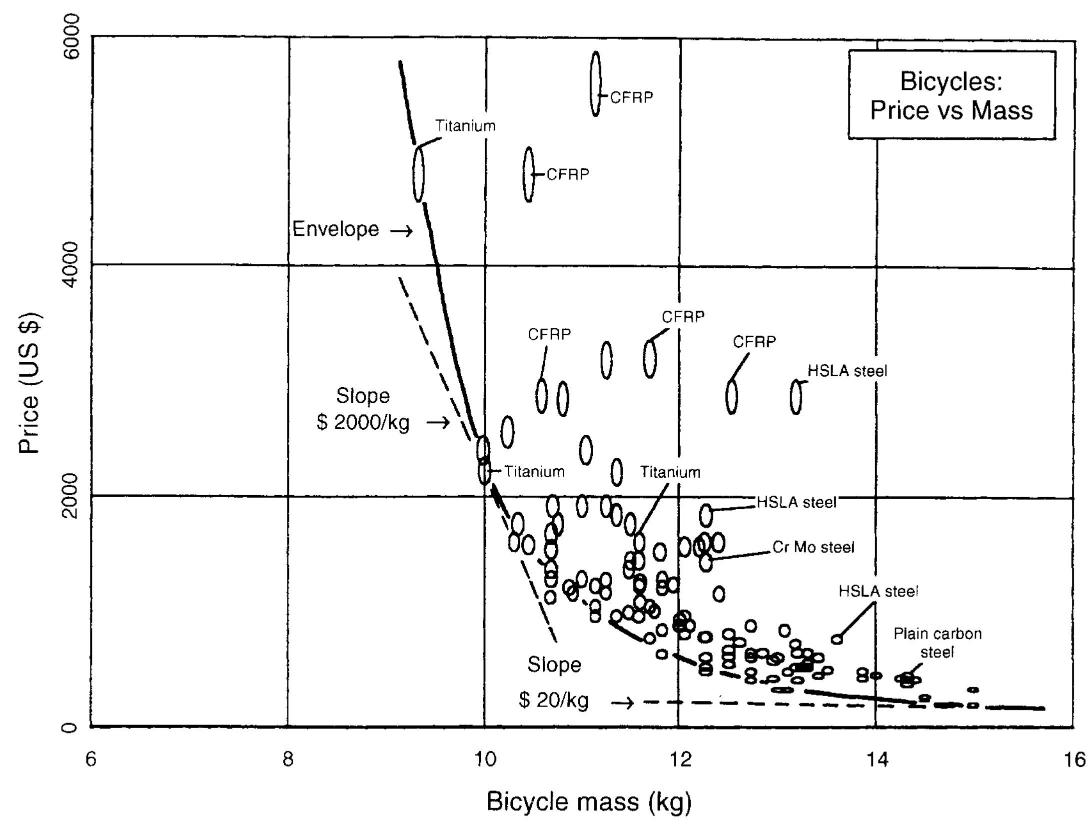

Fig. 9.7 A plot of cost against mass for bicycles. The lower envelope of price is here treated as a contour of constant value. The exchange constant is the slope of a tangent to this curve. It varies with mass.

examples, given earlier, led to the values in Tables 9.3 and 9.4. When — as with bicycles — a range of products has existed long enough for the prices to have stabilized, the exchange constant can be estimated from the appropriate plots, as in Figure 9.7. When this is not so, it may be possible to build up a plot like Figure 9.7 by using interviewing techniques which elicit the change in value that a potential purchaser might associate with a given change in performance*.

Sometimes, however, establishing the exchange constant can be very difficult. An example is that for environmental impact — the damage to the environment caused by manufacture, or use, or disposal of a given product. Minimizing environmental damage could be made an objective, like minimizing cost. Ingenious design can reduce the first without driving the second up too much, but until the exchange constant is defined — by legislation, perhaps, or by necessity — it is difficult for the designer to respond.

An example: value function for mass and cost

We will use the same example as before: the now-very-boring beam. Consider, then, an application for which is sought a material for a *light*, *cheap*, stiff beam (objectives italicized) of prescribed

* For a fuller discussion see de Neufville and Stafford (1971) and Field and de Neufville (1988).

length ℓ and stiffness S. Ignoring shape, its mass, m, is given by equation (9.5) which, repeated, is

$$m = \left(\frac{12\,S}{C_1\,\ell}\right)^{1/2} \ell^3 \left[\frac{\rho}{E^{1/2}}\right] \tag{9.15a}$$

The first objective is to minimize m. The cost of the beam is

$$C = C_m m \tag{9.15b}$$

where C_m is the material cost (in shaped form if necessary) and m is defined by equation (9.15a). The second objective is to minimize C.

To proceed further we need the exchange constant, $E^{\$}$, and this, we know, depends on the application. Given this, we construct a value function, V, following equation (9.14):

$$\boxed{V = E^{\$} m - C} \tag{9.16}$$

Think of it this way: the term $E^{\$}m$ measures the value V to you of a beam of mass m and stiffness S; the term C measures its cost — its exchange constant is simply -1, giving the negative sign. The best choice of material is that with the largest (least negative) value of this function.

Analytical evaluation

Table 9.5 illustrates the use of the method to select a beam with

$$\ell = 1\ \text{m} \qquad S = 10^6\ \text{N/m} \qquad C_1 = 48$$

Substituting these and the material properties shown in the table into equations (9.15a) and (9.15b) gives the values of m and C shown in the table. Forming the value function V of equation (9.16) with $E^{\$} = -1$ \$/kg gives the values shown in the second last column of the table; for this exchange constant, steel wins. The last column shows what happens if $E^{\$} = -100$ \$/kg: the 6061 aluminium alloy maximizes the value function.

Although we have used the values of the design requirements ℓ, S and C_1 to evaluate V, they were not, in fact, necessary to make the selection, which remains unchanged for all values of these variables. This remarkable and useful fact can be understood in the following way. Substituting equations (9.15a) and (9.15b) into (9.16) gives

$$V = (E^{\$} - C_m)m = (E^{\$} - C_m)\left(\frac{12S}{C_1\ell}\right)^{1/2} \ell^3 \left[\frac{\rho}{E^{1/2}}\right] \tag{9.17}$$

Table 9.5 Value functions, V, for two values of exchange constant, $E^{\$}$

Material	ρ *kg/m³*	*E GPa*	C_m^* *\$/kg*	*m, kg*	*t mm*	*C \$*	*V, $E^{\$}$ = −1 \$/kg*	*V, $E^{\$}$ = −100 \$/kg*
1020 Steel	7850	205	0.5	8.7	33	**4.35**	**−13**	−8700
6061 Al	2700	70	1.9	**5.1**	43	9.7	−15	**−5100**
Ti-6-4	4400	115	22	6.5	38	143	−150	−6640

*Cost of material in shape of beam.

This is the quantity we wish to maximize. Rearranging gives

$$\tilde{V} = \frac{V}{\ell^3(12S/C_1\ell)^{1/2}} = E^{\$}\left[\frac{\rho}{E^{1/2}}\right] - \left[\frac{C_m\rho}{E^{1/2}}\right]$$

which we write as

$$\boxed{\tilde{V} = E^{\$}M_1^* - M_2^*} \quad (9.18)$$

Ranking materials by V gives the same order as ranking them by $\tilde{V}$, and this is independent of the values of ℓ, S and C_1. It depends only on $E^{\$}$ and on two material indices $M_1^* = \rho/E^{1/2}$ and $M_2^* = C_m\rho/E^{1/2}$. (These are the reciprocals of indices used earlier; the asterisk on the Ms are a reminder of this.)

Graphical analysis

The graphical method involves a selection chart with axes M_1^* and M_2^*. Consider first the use of the value function to seek a *substitute* for an existing material. The incumbent is material A. On a plot of M_1^* against M_2^* (Figure 9.8) materials which lie below A have a lower value of M_1^*; those which lie to the left have a lower value of M_2^*. It is clear that the materials which lie in the lower left quadrant have lower values of $\tilde{V}$, regardless of the value of $E^{\$}$, and thus are superior to A in performance.

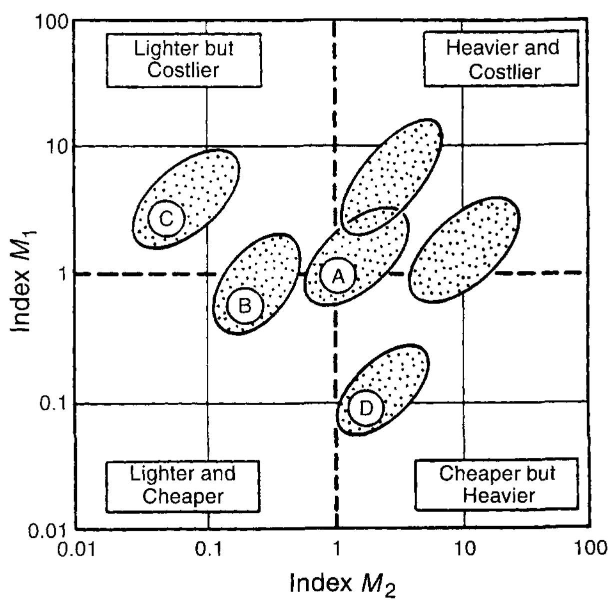

Fig. 9.8 A schematic chart showing two indices, M_1^* measuring cost and M_2^* measuring weight. If the currently used material is A, then B is both cheaper and lighter. The material C is lighter but costs more; D is cheaper but heavier. This selection ignores the trade-off between weight and cost.

That argument is correct, but incomplete. For a given value of the exchange constant, materials in the two neighbouring quadrants can also be viable substitutes. A line of constant $\tilde{V}$, through the material A links materials which have the *same* value of $\tilde{V}$. This line has a slope, found by differentiating equation (9.18), of

$$\left(\frac{\partial M_2^*}{\partial M_1^*}\right)_{\tilde{V}} = E^\$ \qquad (9.19)$$

and this, on linear scales, is a straight line with a slope of $E^\$$. Materials below this line perform better than those on or above it — and this now includes materials in the neighbouring quadrants.

In practice, the ranges of M_1^* and M_2^* are large, and it becomes more attractive to use logarithmic scales. Then the line becomes curved. Figure 9.9 shows lines of constant $\tilde{V}$ for three values of $E^\$$; it is now apparent that the straight, horizontal and vertical lines on the figure are the extremes, corresponding to $E^\$ = 0$ and $E^\$ = \pm\infty$. As Figure 9.9 shows, a low value of $E^\$$ makes material D a better choice than A, and a high one makes material C a better choice, although neither lie in the bottom left quadrant.

For *selection* we require contours for different values of $\tilde{V}$ (equation (9.18)) not just for the one on which material A sits. Figure 9.10 shows, for two values of $E^\$$, the way the contours look. They all have the same shape; as $\tilde{V}$ decreases, they move downwards along the line

$$\tilde{M}_2 = E^\$\tilde{M}_1 - \tilde{V}$$

The best choice of material, for a given $E^\$$, is that with the lowest $\tilde{V}$. Note that the absolute value of $\tilde{V}$ is unimportant, either for selection or for substitution; it is only the relative value that is

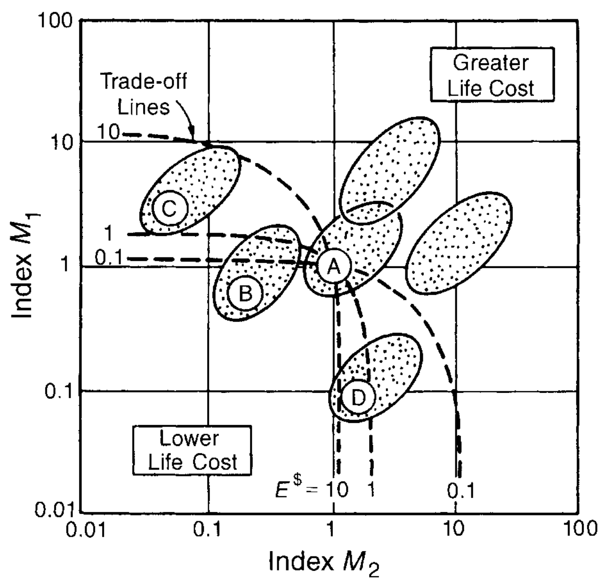

Fig. 9.9 The proper comparison of A with competing materials is made by constructing a value-function, $\tilde{V}$, which combines the cost of material with that associated with weight, using the weight–cost exchange constant $E^\$$. When $E^\$ = 10$ \$/kg, C is a better choice than A; when $E^\$ = 0.1$ \$/kg, D is better than A.

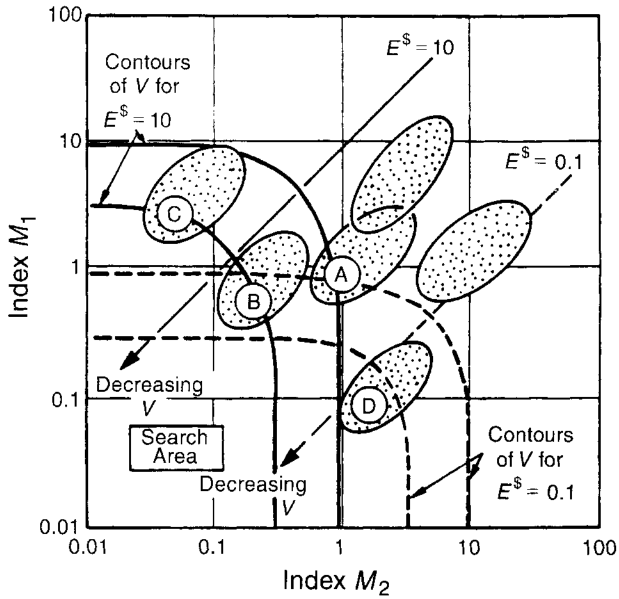

Fig. 9.10 Selection to minimize total life-cost, using the exchange constant method. The contours show the value-function, $\tilde{V}$, for two values of $E^\$$.

required. Thus the constant $\ell^3(12S/C_1\ell)^{1/2}$ in equation (9.15) can be dropped; and the only matter of importance is to ensure that M_1^*, M_2^* and $E^\$$ are expressed in consistent units.

Described in the abstract, as here, these methods sound a little complicated. The case studies of Chapter 10 will illustrate how they work.

9.7 Summary and conclusions

The method of material indices allows a simple, transparent procedure for selecting materials to meet an objective (like minimizing component weight) while meeting a constraint (safely carrying a design load, for instance). But most designs have several constraints, and it is usual that the selection is driven by divergent objectives.

Judgement can be used to rank the importance of the competing constraints and objectives, and this is often the simplest route. To do this, an index is derived for the most important of these, which is then used to select the first, broad, subset of materials. The members of the subset are now examined for their ability to satisfy the remaining constraints and objectives. Weight-factors or fuzzy logic put the judgement on a more formal footing, but can also obscure its consequences.

Judgement can, sometimes, be replaced by analysis. For multiple constraints, this is done by identifying the active constraint and basing the design on this. The procedure can be made graphical by deriving coupling equations which link the material indices; then simple constructions on material selection charts with the indices as axes identify unambiguously the subset of materials which maximize performance while meeting all the constraints. Compound objectives require consideration of the exchange constant; it allows both objectives to be expressed in the same units (usually cost) and the formulation of a value-function V which guides material choice. Here, too, simple constructions on charts with material indices as axes allow the optimum subset of materials to be selected.

When multiple constraints operate, or a compound objective is involved, the best choice of material is far from obvious, and can be counter-intuitive. It is here that the methods developed have real power.

9.8 Further reading

Multiple constraints and compound objectives

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